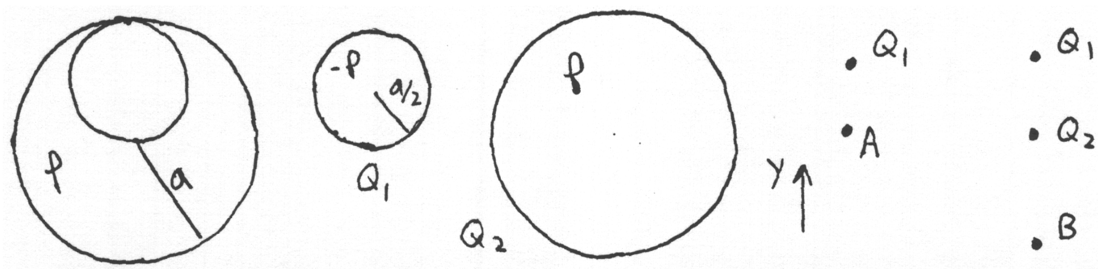
University of California, Berkeley
Physics H7B Spring 1999 (*Strovink*)

SOLUTION TO PROBLEM SET 5
Solutions by P. Pebler

1 *Purcell 1.16* *A sphere of radius a was filled with positive charge of uniform density ρ. Then a smaller sphere of radius $a/2$ was carved out, as shown, and left empty. What are the direction and magnitude of the electric field at points A and B?*



The key is to consider the given distribution as a superposition of the two distributions at right. The electric field will be the sum of the contributions from these two spheres, which are easy to evaluate. For points outside these spheres, we may treat them as point charges lying at their centers. The charges are

$$Q_1 = \frac{4}{3}\pi\left(\frac{a}{2}\right)^3(-\rho) = -\frac{\pi a^3\rho}{6},$$

$$Q_2 = \frac{4}{3}\pi a^3\rho.$$

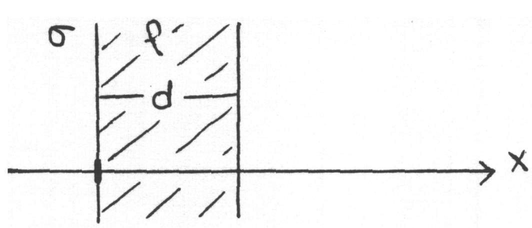
Consider the point A. The contribution from sphere 2 is zero since there is no field at the center of a spherical distribution. The point A lies outside the sphere 1.

$$\mathbf{E}_A = \frac{Q_1}{(a/2)^2}(-\hat{\mathbf{y}}) = \frac{2\pi}{3}a\rho\,\hat{\mathbf{y}}$$

Point B lies outside both spheres.

$$\mathbf{E}_B = \frac{Q_2}{a^2}(-\hat{\mathbf{y}}) + \frac{Q_1}{(3a/2)^2}(-\hat{\mathbf{y}}) = -\frac{4\pi a^3\rho}{3a^2}\,\hat{\mathbf{y}} + \frac{2\pi a^3\rho}{27a^2}\,\hat{\mathbf{y}} = -\frac{34}{27}\pi a\rho\,\hat{\mathbf{y}}$$

2 *Purcell 1.19* *An infinite plane has a uniform surface charge distribution σ on its surface. Adjacent to it is an infinite parallel layer of charge of thickness d and uniform volume charge density ρ. All charges are fixed. Find the electric field everywhere.*



The contribution due to the surface charge has magnitude $2\pi|\sigma|$ and points away from or towards the surface depending on the sign of σ. To deal with the volume charge, we can treat it as a stack of very thin layers of charge and treat these layers as surface charges. We could add up all the contributions from these infinitesimal layers by integrating. However, since the field from an infinite plane of charge does not depend on how far away you are, the contribution from each layer will be the same. So we will get the same answer by assuming the finite volume charge layer to be a surface with surface charge $\rho\, t$ where t is the thickness of the layer in question. For $x < 0$, everything pushes to the left.

$$\mathbf{E} = 2\pi\sigma(-\hat{\mathbf{x}}) + 2\pi\rho\, d(-\hat{\mathbf{x}}) = -2\pi(\sigma + \rho\, d)\,\hat{\mathbf{x}} \qquad x < 0$$

Likewise,

$$\mathbf{E} = 2\pi(\sigma + \rho\, d)\,\hat{\mathbf{x}} \qquad x \geq d.$$

For the region $0 < x < d$, the volume layer is split into two. We can thing of the right side as a single surface with surface charge $\rho(d - x)$ pushing to the left, and the left side as a surface charge ρx pushing to the right.

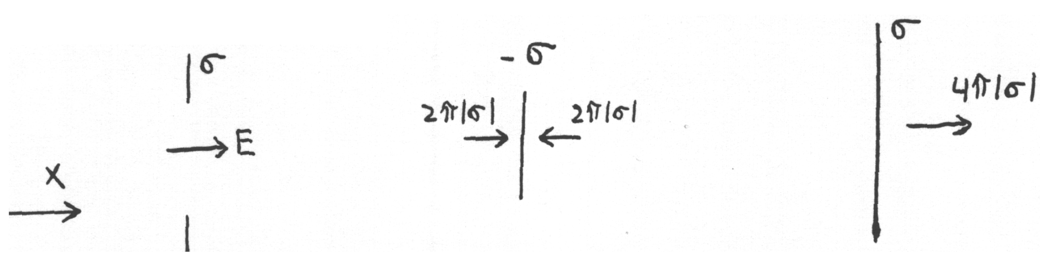
$$\mathbf{E} = 2\pi\rho\,\hat{\mathbf{x}} + 2\pi\rho\, x\,\hat{\mathbf{x}} - 2\pi\rho(d - x)\,\hat{\mathbf{x}} = (2\pi\sigma + 4\pi\rho\, x - 2\pi\rho\, d)\,\hat{\mathbf{x}} \qquad 0 < x < d$$

If we wished to consider the plane $x = 0$, we could say that the surface charge σ contributes nothing.

$$\mathbf{E} = -2\pi\rho\, d\,\hat{\mathbf{x}} \qquad x = 0$$

Notice that there is a discontinuity of $4\pi\sigma$ as we pass through zero. This is always the case for idealized surface charges. There is no discontinuity at $x = d$ however.

3 *Purcell 1.29 A spherical shell of charge of radius a and surface charge density σ is missing a small, approximately circular, piece of "radius" $b \ll a$. What is the direction and magnitude of the field at the midpoint of the aperture?*



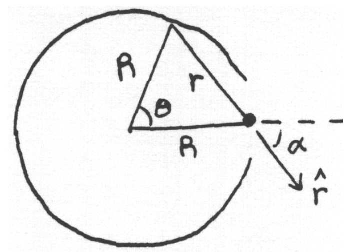
The picture above assumes for simplicity that $\sigma > 0$.

As a zero order approximation, we may consider the missing circle as infinitesimal. The field at left can be viewed as a superposition of the two distributions at right. We temporarily ignore points exactly at the surface. By considering the field to the right or to the left we find

$$\mathbf{E} = 4\pi\sigma\,\hat{\mathbf{x}} + 2\pi(-\sigma)\,\hat{\mathbf{x}} = \mathbf{0} + 2\pi(-\sigma)(-\hat{\mathbf{x}}) = 2\pi\sigma\,\hat{\mathbf{x}},$$

for the field everywhere except at the surface. But for the distribution with the circle missing, there can be no discontinuity when passing through the hole, so the field directly at the surface is also $2\pi\sigma\,\hat{\mathbf{x}}$.

This should be a good approximation when $b \ll a$. But for a finite missing piece, this will not be the exact answer even at the center. To find the contributions of higher order in b/a we can integrate. This is actually not too bad.



From symmetry considerations, we know the field is radial in the center of the aperture.

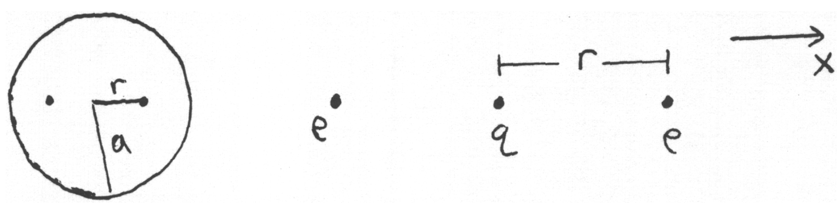
$$\begin{aligned} dE_r &= \frac{\sigma da}{r^2}\cos\alpha = \frac{\sigma R^2 2\pi\, d\theta}{2R^2(1-\cos\theta)}\cos(90-\theta/2) \\ &= \pi\sigma\frac{\sin\theta\sin\frac{\theta}{2}(1+\cos\theta)}{\sin^2\theta}\,d\theta = \pi\sigma\frac{(1+\cos\theta)}{2\cos\frac{\theta}{2}}\,d\theta = \pi\sigma\cos\frac{\theta}{2}\,d\theta \end{aligned}$$

$$E_r = \pi\sigma\int_{\theta_o}^{\pi}\cos\frac{\theta}{2}\,d\theta = 2\pi\sigma\left(1-\sin\frac{\theta_o}{2}\right)$$

We can take the initial angle θ_o to be b/R.

$$E_r = 2\pi\sigma\left[1-\left(\frac{b}{2R}-\frac{1}{3!}\left(\frac{b}{2R}\right)^3+\dots\right)\right]$$

4 *Purcell 1.33* *Imagine a sphere of radius a filled with negative charge −2e of uniform density. Imbed in this jelly of negative charge two protons and assume that in spite of their presence the negative charge remains uniform. Where must the protons be located so that the force on each of them is zero?*



The forces on the protons from each other will be equal and opposite. Therefore, the forces on them from the negative charge distribution must be equal and opposite also. This requires that they lie on a line through the center and are equidistant from the center. The force on each proton at radius r from the negative charge will be proportional to the amount of negative charge lying inside a sphere of radius r. For purposes of finding the electric field, we may treat all of this charge as if it were a point charge sitting in the center. We ignore all negative charge outside the radius of the proton positions. The negative charge inside the radius r is

$$q = -\frac{r^3}{a^3}2e.$$

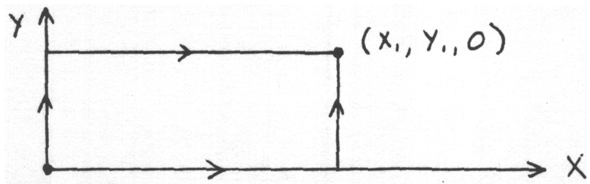
The force on the right proton must be zero.

$$\mathbf{F} = \frac{e^2}{(2r)^2}\hat{\mathbf{x}} + \frac{e(-2er^3/a^3)}{r^2}\hat{\mathbf{x}} = \frac{e^2}{(2r)^2}\left(1-8\frac{r^3}{a^3}\right)\hat{\mathbf{x}} = \mathbf{0} \qquad r = \frac{a}{2}$$

5 *Purcell 2.1* *The vector function*

$$E_x = 6xy \qquad E_y = 3x^2-3y^2 \qquad E_z = 0$$

represents a possible electrostatic field. Calculate the line integral of $\mathbf{E}$ *from the point* $(0,0,0)$ *to the point* $(x_1, y_1, 0)$ *along the path which runs straight from* $(0,0,0)$ *to* $(x_1, 0, 0)$ *and thence to* $(x_1, y_1, 0)$*. Make a similar calculation for the path which runs along the other two sides of the rectangle, via the point* $(0, y_1, 0)$*. Now you have the a potential function* $\phi(x,y,z)$*. Take the gradient of this function and see that you get back the components of the given field.*



We take the first path in two parts. While moving along the x axis we have $d\mathbf{s} = dx\,\hat{\mathbf{x}}$ so that $\mathbf{E}\cdot d\mathbf{s} = E_x\,dx$ and while moving up parallel to the y axis we have $d\mathbf{s} = dy\,\hat{\mathbf{y}}$ and $\mathbf{E}\cdot d\mathbf{s} = E_y$.

$$\int \mathbf{E}\cdot d\mathbf{s} = \int_0^{x_1} E_x\,dx + \int_0^{y_1} E_y\,dy = \int_0^{x_1} 6xy\,dx + \int_0^{y_1} (3x^2 - 3y^2)\,dy$$

When integrating along the x axis, y has the constant value $y = 0$ which we plug in to the first integral. Along the second part of the path, $x = x_1$.

$$\int \mathbf{E}\cdot d\mathbf{s} = 0 + \int_0^{y_1} (3x_1^2 - 3y^2)\,dy = 3x_1^2 y_1 - y_1^3$$

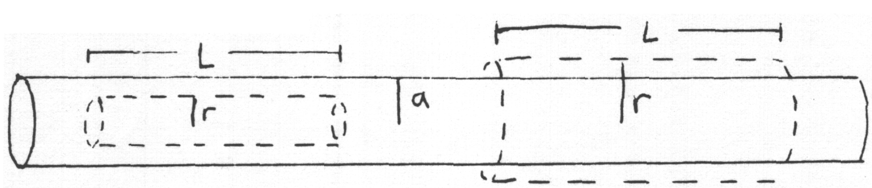
We do the same thing along the second path.

$$\int \mathbf{E}\cdot d\mathbf{s} = \int_0^{y_1} (3(0)^2 - 3y^2)\,dy + \int_0^{x_1} 6xy_1\,dx = -y_1^3 + 3x_1^2 y_1$$

$$\phi(x,y,z) = 3x^2y - y^3$$

$$\nabla\phi(x,y,z) = \frac{\partial}{\partial x}(3x^2y - y^3)\,\hat{\mathbf{x}} + \frac{\partial}{\partial y}(3x^2y - y^3)\,\hat{\mathbf{y}} + \frac{\partial}{\partial z}(3x^2y - y^3)\,\hat{\mathbf{z}} = 6xy\,\hat{\mathbf{x}} + (3x^2 - 3y^2)\,\hat{\mathbf{y}}$$

6 *Purcell 2.8* *Consider an infinitely long cylinder of radius a and uniform charge density* ρ*. Use Gauss's law to find the electric field. Find the potential* ϕ *as a function of* r*, both inside and outside the cylinder, taking* $\phi = 0$ *at* $r = 0$*.*



Our Gaussian surface both inside and outside the cylinder will be a cylinder of length L. Inside we have

$$\int \mathbf{E}\cdot d\mathbf{a} = E_r 2\pi r L = 4\pi Q_{enc} = 4\pi\pi r^2 L\rho,$$

$$\mathbf{E} = 2\pi\rho\, r\,\hat{\mathbf{r}} \qquad r < a.$$

Outside,

$$\int \mathbf{E}\cdot d\mathbf{a} = E_r 2\pi r L = 4\pi Q_{enc} = 4\pi\pi a^2 L\rho,$$

$$\mathbf{E} = \frac{2\pi\rho a^2}{r}\,\hat{\mathbf{r}} \qquad\qquad r \geq a.$$

To find the potential, we integrate radially outward from the center so that $d\mathbf{s} = dr\,\hat{\mathbf{r}}$.

$$\phi(r) = \phi(0) - \int \mathbf{E}\cdot d\mathbf{s} = 0 - \int_0^r 2\pi\rho\, r'\,dr' = -\pi r^2\rho \qquad\qquad r \leq a$$

For points outside the cylinder,

$$\phi(r) = \phi(a) - \int_a^r \frac{2\pi\rho a^2}{r'}\,dr' = -\pi a^2\rho - 2\pi\rho\, a^2 \ln\frac{r}{a} \qquad\qquad r > a.$$

7 *Purcell 2.19 Two metal spheres of radii R_1 and R_2 are quite far apart compared with these radii. Given a total amount of charge Q, how should it be divided so as to make the potential energy of the resulting charge distribution as small as possible? Assume that any charge put on one of the spheres distributes itself uniformly over the sphere. Show that with that division the potential difference between the spheres is zero.*

Because the spheres are far apart, the energy will be essentially due to the energy of each sphere. We may assume that the charge on each sphere is uniformly distributed if the other sphere is very far away. To find this energy we can use the standard formula adapted to surface charge,

$$U = \frac{1}{2}\int \sigma\phi\,da.$$

The potential ϕ just outside a uniformly charged sphere is q/r and because the potential is continuous, this is also the potential at the surface. Then,

$$U = \frac{1}{2}\int \frac{q}{4\pi r^2}\frac{q}{r} r^2 \sin\theta\, d\phi d\theta = \frac{1}{2}\frac{q^2}{r}.$$

Breaking up the charge into q and $Q - q$,

$$U = \frac{q^2}{2R_1} + \frac{(Q-q)^2}{2R_2}.$$

If the minimum energy is obtained with $q = q_o$,

$$\frac{dU}{dq}(q_o) = \frac{q_o}{R_1} - \frac{(Q-q_o)}{R_2} = 0,$$

$$\frac{q_o}{R_1} = \frac{Q-q_o}{R_2}.$$

But these are just the potentials at both spheres.

8 *Purcell 2.20 As a distribution of electric charge, the gold nucleus can be described as a sphere of radius 6×10^{-13} cm with a charge $Q = 79e$ distributed fairly uniformly through its interior. What is the potential ϕ_o at the center of the nucleus, expressed in megavolts?*

For a uniformly charged sphere of radius a,

$$\mathbf{E} = \frac{Q}{r^2}\,\hat{\mathbf{r}} \qquad\qquad r > a,$$

$$\mathbf{E} = \frac{Qr}{a^3}\,\hat{\mathbf{r}} \qquad\qquad r \le a.$$

Since the potential is zero at infinity, the potential at any point P is

$$\phi(P) = -\int_{\infty}^{P} \mathbf{E}\cdot d\mathbf{s}.$$

We can make the path of integration come radial straight in. If the point P has $r < a$,

$$\phi(r) = -\int_{\infty}^{r} E_{r'}\,dr' = -\int_{\infty}^{a} \frac{Q}{r'^2}\,dr' - \int_{a}^{r} \frac{Qr'}{a^2}\,dr' = \frac{Q}{a} - \frac{Qr^2}{2a^3} + \frac{Q}{2a}$$

In SI units,

$$\phi(0) = \frac{1}{4\pi\epsilon_o}\frac{3Q}{2a} = \frac{3\cdot 79(1.6\times 10^{-19}\ C)}{4\pi(8.85\times 10^{-12}\ C^2/Nm^2)2(6\times 19^{-15}\ m)} = 28.4\ megavolts.$$